# Derivation of the Euler-Lagrange Equation using Newton's 2nd Law

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Given a set of generalized coordinates $q_i$ compatible with constraints, note that $\frac{\partial \vec{r}}{\partial q_i}$ is always pointing into a direction into which movement is not forbidden by the constraints. Let's project Newton's 2nd law into these directions:

$$\vec{F} = m\ddot{\vec{r}}$$
$$\vec{F} \cdot \frac{\partial \vec{r}}{\partial q_i} = m\ddot{\vec{r}} \cdot \frac{\partial \vec{r}}{\partial q_i}$$

The way $F$ is used in the first equation, it needs to contain the constraining forces in order to give the correct dynamics. The nice thing about the projection into the allowed subspace is that these constraining forces are eliminated out of the projected equations because they are perpendicular to the allowed subspace. Thus, after the projection we can neglect the constraining forces in $F$ and can also safely replace $F$ with $-\nabla V$ despite the fact that the potential $V$ doesn't know about constraints and we still obtain the correct dynamics. With $\nabla V \cdot \frac{\partial \vec{r}}{\partial q_i} = \frac{\partial V}{\partial q_i}$ we rewrite the left side of the equation above:

$$-\frac{\partial V}{\partial q_i} = m\ddot{\vec{r}} \cdot \frac{\partial \vec{r}}{\partial q_i}$$

Rewriting the right side in terms of the kinetic Energy $T$ leads to the Euler-Lagrange equation:

$$\begin{aligned}
-\frac{\partial V}{\partial q_i} &= m\frac{d}{dt}\left(\dot{\vec{r}}\right) \cdot \frac{\partial \vec{r}}{\partial q_i} \\
&= m\left(\frac{d}{dt}\left(\dot{\vec{r}} \cdot \frac{\partial \vec{r}}{\partial q_i}\right) - \left(\dot{\vec{r}} \cdot \frac{d}{dt}\frac{\partial \vec{r}}{\partial q_i}\right)\right) \qquad \Big| \text{ with } \frac{\partial \vec{r}}{\partial q_i} = \frac{\partial \dot{\vec{r}}}{\partial \dot{q}_i}{}^{1} \\
&= m\left(\frac{d}{dt}\left(\dot{\vec{r}} \cdot \frac{\partial \dot{\vec{r}}}{\partial \dot{q}_i}\right) - \left(\dot{\vec{r}} \cdot \frac{\partial \dot{\vec{r}}}{\partial q_i}\right)\right) \\
&= m\left(\frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\frac{\dot{\vec{r}}^2}{2} - \frac{\partial}{\partial q_i}\frac{\dot{\vec{r}}^2}{2}\right) \\
&= \frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}T - \frac{\partial}{\partial q_i}T
\end{aligned}$$

With $\mathcal{L} = T - V$ and $\frac{\partial V(q_i)}{\partial \dot{q}_i} = 0$ we get:

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0$$

[1] $\frac{\partial \dot{\vec{r}}}{\partial \dot{q}_i} = \frac{\partial}{\partial \dot{q}_i}\left(\frac{d\vec{r}}{dt}\right) = \frac{\partial}{\partial \dot{q}_i}\left(\frac{\partial \vec{r}}{\partial q_i}\dot{q}_i\right) = \frac{\partial \vec{r}}{\partial q_i}$